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The behaviour of a stable salinity gradient heated from below

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The motions resulting when a linear, stable salt gradient is heated uniformly and at a steady rate from below are investigated theoretically and by laboratory experiment. A convecting, growing layer is first formed whose depth, temperature and salinity differences from the fluid above, are all increasing as $t^{\frac{1}{2}}$. The way in which these quantities depend on the salinity gradient and heating rate is also predicted, and verified experimentally. A stability criterion is then developed which describes the breakdown of the diffusive boundary layer ahead of the advancing front, and leads to an expression for the thickness of the bottom layer when a second layer forms above it. The predicted form of dependence of layer thickness on the given parameters is again borne out by the experiments.

1. Introduction

In an earlier paper (Turner & Stommel 1964) a qualitative description was given of the motions which result when a linearly stratified salt solution is heated uniformly from below. Convective stirring first produces a layer at the bottom of the containing vessel which is well mixed in both heat and salt, and which grows by incorporating fluid from above. This layer does not continue to deepen indefinitely, however; at some point a second layer forms on top of it, and this behaves in the same way. In time, many such layers can form, with sharp interfaces separating the turbulent convecting regions, and these interfaces stay in nearly fixed positions until the bottom one disappears as the lowest two layers merge. The system can be maintained in this state because of the very different molecular diffusivities of heat and salt: heat can more easily escape from the top of a layer to cause a convective overturning above, while the salt mostly stays behind and preserves a net stable density difference across the interface.

There is now an increasing volume of observational evidence to suggest that a step-like structure is characteristic of the ocean and also of some lakes. Instead of smooth gradients of temperature and salinity, one often observes well-mixed layers separated by relatively sharp interfaces. In some cases at least, the distributions of salinity and temperature are similar to those just described for the laboratory experiment (see, for example, the observations at the bottom of the Red Sea described by Swallow & Crease (1965), and those in an Antarctic lake reported by Hoare (1966)). The hope that a closer study of the small-scale laboratory models will lead to a quantitative understanding of such phenomena provides the immediate incentive for the present work.

The only part of this problem which has so far been treated theoretically is the initial behaviour of a stable salinity gradient when a destabilizing temperature gradient is applied to it. Veronis (1965) and Sani (1965) showed that the onset of convection should occur in an overstable mode, rather than as an exponentially growing motion (as in the ordinary convection problem with heat alone). Although the boundary conditions assumed are not exactly right for the transient case being considered here, Shirtcliffe (1967) has confirmed experimentally that an overstable oscillation is the relevant mode of breakdown. The initial oscillatory behaviour will be documented further in this paper, but most of the experiments reported here (and all those described earlier by Turner & Stommel) relate to the later stages of an experiment when the first layer has already formed. That is, the heating rate will usually be so great that convection can be supposed to begin in a time short compared to the timescale of growth of the bottom layer, and the earliest stages of formation of the first layer will be ignored.

It was recognized from the beginning that one of the main problems posed by these experiments with a heated salinity gradient is the relation of the layer depths to the magnitude of the salinity gradient and the heating rate. There is, however, a complicating factor which led to the deferment of this part of the study. It was found that side-wall heating can also produce layering in a stable salinity gradient, as convection cells are propagated horizontally into the interior. A very small temperature difference between the fluid and the container can thereby produce interfaces which are unrelated to the heating from below, but which can determine the position where a new layer begins to form. For this reason, the first quantitative experiments (Turner 1965) were of a different kind. Instead of being produced from a gradient, two uniform layers were set up directly with a density interface between them, and the transports of heat and salt across this interface measured as a function of the temperature and salinity differences. The information thus obtained represents part of that necessary for a full understanding of the phenomenon, but some quantities, such as the 'eddy transport coefficients' relating the fluxes to the mean gradients, cannot be defined until the depths of the layers are known.

The present paper returns to the original problem, the establishment of layers by heating from below. The anticipated difficulties due to side-wall heating have been minimized by concentrating only on the period of growth of the *first* layer, up until the time a second layer forms above it. The work to be described falls into two nearly separate parts. First, theoretical arguments will be presented which relate the rate of advance of the top of the first layer to the salinity gradient and heating rate (both of these being regarded as constant) and these relations will be tested quantitatively through laboratory experiments. Secondly, the development of a temperature boundary layer in the stationary fluid ahead of the advancing interface will be investigated. A Rayleigh number criterion for the breakdown of the second layer will be suggested, based on the initial properties and the developing temperature distribution. This leads to an explicit form for the dependence of the maximum depth of the bottom layer on the salinity gradient and heating rate, and this too is tested by experiment.

2. The growth of the bottom layer

2.1. *Theory*

In this and all subsequent sections it will be assumed that the initial state is one of linear salinity gradient dS/dz in a deep layer which is at rest above a solid, heat conducting bottom. The temperature is uniform, until at time $t = 0$ a constant heat flux is applied through the bottom and is maintained at H cal/cm^2/sec at all later times t. This heating soon produces a well-stirred growing layer as discussed earlier; both temperature and salinity are supposed uniform in this layer and it is assumed that there is an abrupt step of both temperature and salinity at its top. That is, the molecular diffusion of heat and salt ahead of the growing convective layer is neglected in this section, though it will be considered explicitly later.

From the heat and salinity balances in the bottom layer of depth h it follows immediately that

$$Ht = -\rho ch\Delta T \tag{1}$$

and

$$\Delta S = \tfrac{1}{2}h(dS/dz), \tag{2}$$

where ΔT and ΔS are the temperature and salinity steps at the top of the well-mixed layer at time t (these are both negative, as is dS/dz if z is taken to be positive upwards), c is the specific heat, and ρ is a characteristic density (deviations from this are supposed to be small).

A relation between ΔT and ΔS may be obtained by making one of several possible assumptions about the energy balance in the bottom layer. Although this will later be rejected as unrealistic, a useful limiting case is obtained by assuming first that all work done by heating from below is used to change the salinity distribution. The equivalent of this assumption was justified by Ball (1960) in a discussion of the change in inversion height due to surface heating, and was also used by Turner & Kraus (1967) in a study of a laboratory experiment related to the seasonal thermocline. The difference in the present context is that two components now contribute to the potential energy, and their effects will be kept distinct. This statement of conservation of energy implies that the potential energy (relative to the bottom of the tank say) of the original linear salinity gradient, and the final step distribution of salinity and temperature, must be the same. Now the change of potential energy per unit mass due to the addition of heat is $\frac{1}{2}g\alpha h^2\Delta T$, where α is the coefficient of expansion. The change due to the stirring of the bottom of the salt distribution to make it uniform is $\frac{1}{12}g\beta h^3 dS/dz$ where β is the corresponding coefficient for salt, i.e. the proportional change in density for unit salinity change. It follows that

$$\alpha\Delta T = -\tfrac{1}{6}\beta h(dS/dz) \tag{3}$$

or comparing with (2)

$$\alpha\Delta T = -\tfrac{1}{3}\beta\Delta S. \tag{4}$$

(Note that α is a negative quantity and β is positive.) Thus with this assumption about the energy balance, the temperature and salinity differences remain proportional to one another as the layer grows, and there is a net stable density step at its top. The corresponding distributions of properties in this case are sketched in figure 1*a*.

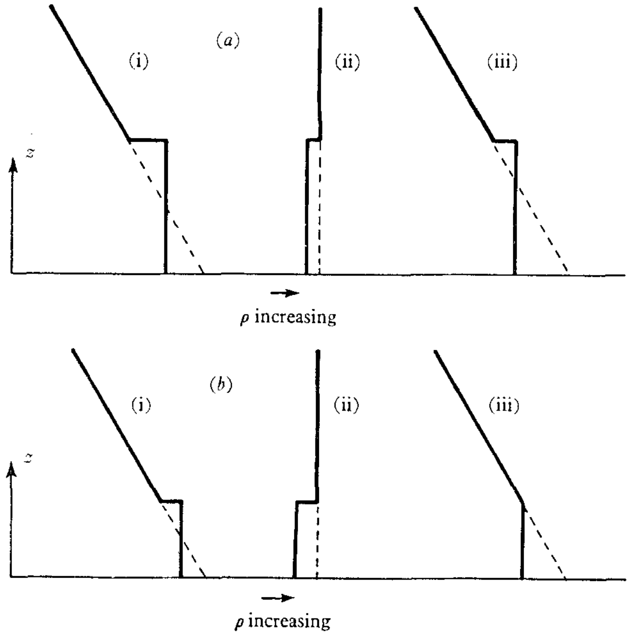

FIGURE 1. The distributions of (i) density due to salinity, (ii) density due to temperature, (iii) the net density, implied by the two assumptions examined in the text: (*a*) all the potential energy released in the temperature field is used to redistribute the salt; (*b*) the top of the mixed layer is marginally stable. Heating has been assumed at the same rate and for the same time in the two cases.

The limiting energy condition assumed above can clearly not be realized in practice. Dissipation has been neglected, and the fact that kinetic energy is released throughout the layer (not just at the top) has been ignored. At the other extreme, one can assume that the layer depth is limited not by the process of entrainment across a density step, as implied above, but because the net density gradient is always just unstable at its top. This implies instead of (4) that

$$\alpha \Delta T = -\beta \Delta S, \tag{5}$$

i.e. that the density steps due to temperature and salinity are equal and opposite (as shown in figure 1*b*). Because the condition (5) is found to be closely satisfied in the laboratory experiments reported later, the analysis will be continued using this second assumption.

The equivalent of these two assumptions has recently been used by Lilly (1967) in a treatment of radiating cloud layers under a strong atmospheric inversion. He found that it was impossible to decide between them on the basis of the available observations, and certainly no measurements in the ocean are yet accurate enough for this purpose. It therefore remains an open question whether, on a larger scale, a faster growth of a layer and a density step could occur because some fraction of the kinetic energy due to heating is used for entrainment. For our present purpose this is not important; provided only that the fraction used in this way is assumed to be *constant*, the functional form of all the relations to be obtained will remain the same, since ΔT and ΔS will still be proportional to one another.

From (2) and (5) it can be seen that h also is proportional to ΔT or ΔS during the growth of a layer, with the constant of proportionality depending only on the initial salinity gradient (cf. equation (3)). Using these with (1) gives now the dependence of h, ΔT and ΔS separately on time:

$$h = H_*^{\frac{1}{2}} S_*^{-\frac{1}{2}} t^{\frac{1}{2}} \tag{6}$$

and

$$g\alpha\Delta T = -g\beta\Delta S = H_*^{\frac{1}{2}} S_*^{\frac{1}{2}} t^{\frac{1}{2}}, \tag{7}$$

where

$$H_* = -g\alpha H/\rho c \quad \text{and} \quad S_* = -\tfrac{1}{2}g\beta(dS/dz). \tag{8}$$

These parameters are defined in this way (including g the acceleration due to gravity) so that they specify the physically relevant buoyancy flux and the corresponding dynamical effect of the density gradient.

2.2. *The experimental technique*

The equipment used is very simple, but it must be described briefly before the results of the quantitative tests of the above relations are presented. All the experiments were carried out in a plastic circular cylinder, 29 cm in diameter and about 25 cm deep. It was fitted with an aluminium bottom 6 mm thick; below this was an air space 5 cm deep, which was well insulated below and round its edge. A regular array of heater wires was strung in the air space on insulating supports, to provide heating which was fairly uniform over the base (although extreme precautions were not taken to make it so). The rate of heating for various settings of a rheostat supplying the voltage to the heater elements was measured directly in a preliminary series of experiments, by recording the rate of change of temperature in a well-stirred layer of fresh water in the cylinder.

The desired linear salinity gradients were set up using a method described by Oster (1965), which is sketched in figure 2. Two tanks of the same shape, containing equal depths of salt solution of different densities and at room temperature were joined by a wide pipe. The liquid in the lighter tank was vigorously stirred while its gradually varying contents were slowly drained into the bottom of the experimental cylinder. It may easily be shown that if the rate of flow out of the stirred tank is just twice that into it, and stirring is complete (conditions which were closely satisfied in our case) then the resulting density gradient will be linear. Knowing the weight of salt, the volume of water used and the dimensions of the experimental tank, the salinity gradient can be deduced; and this was the method used to compute the gradient in the experiments, after a few runs in which it was checked directly by withdrawing and weighing samples.

When the salinity gradient was set up, heating was begun as soon as possible after the filling pipe was withdrawn and the motions had died away, so that it was reasonable to assume that the linear gradient extended to the bottom. In some of the runs, a 16 cm diameter 'guard ring' of Perspex was used to isolate the centre of the tank. No difference could be detected between the experiments carried out with and without this, supporting the assumption that heating through the side walls was unimportant. The quantities recorded as functions of time were the bottom-layer temperature (usually with a bulb thermometer held 1 cm above the bottom of the tank, but occasionally with a faster responding thermistor) and the mixed layer depth (estimated visually by averaging by eye, with a few crystals of dye added to mark the convecting layer). Experiments were conducted over a range of conditions in which there was a tenfold variation of both density gradient and heating rate; typical values of these parameters were $\rho^{-1}(d\rho/dz) = 10^{-3}\,\mathrm{cm}^{-1}$ and $H = 10^{-1}\,\mathrm{cal/cm^2/sec}$.

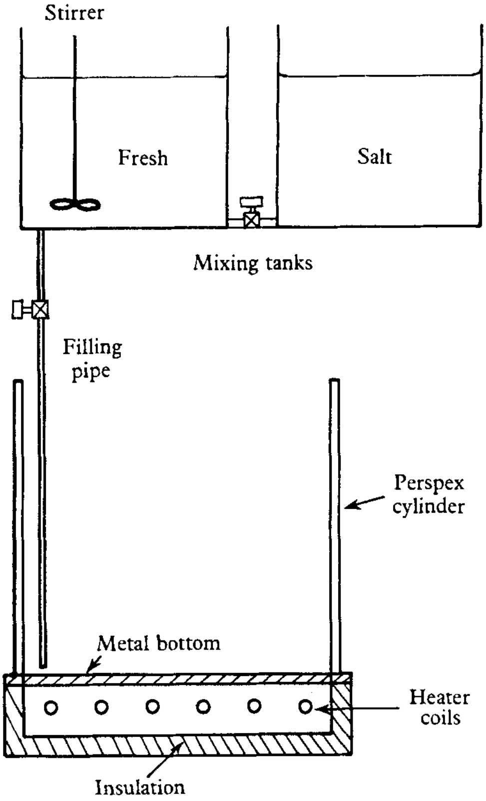

FIGURE 2. A sketch of the experimental equipment, showing the cylinder, the method of heating from below, and the device used for filling with a linear salinity gradient.

2.3. *The experimental results*

The relations (6) and (7), or the more basic assumptions from which they are derived, can be tested by plotting the results in various ways, and not all the possible variations will be covered here. First, it will be shown that the dependence on time is of the indicated functional form, using an individual run as an example. Pairs of measured parameters from many experiments will then be plotted in a form which allows a numerical evaluation of the constants of proportionality, and hence a quantitative test of the assumptions.

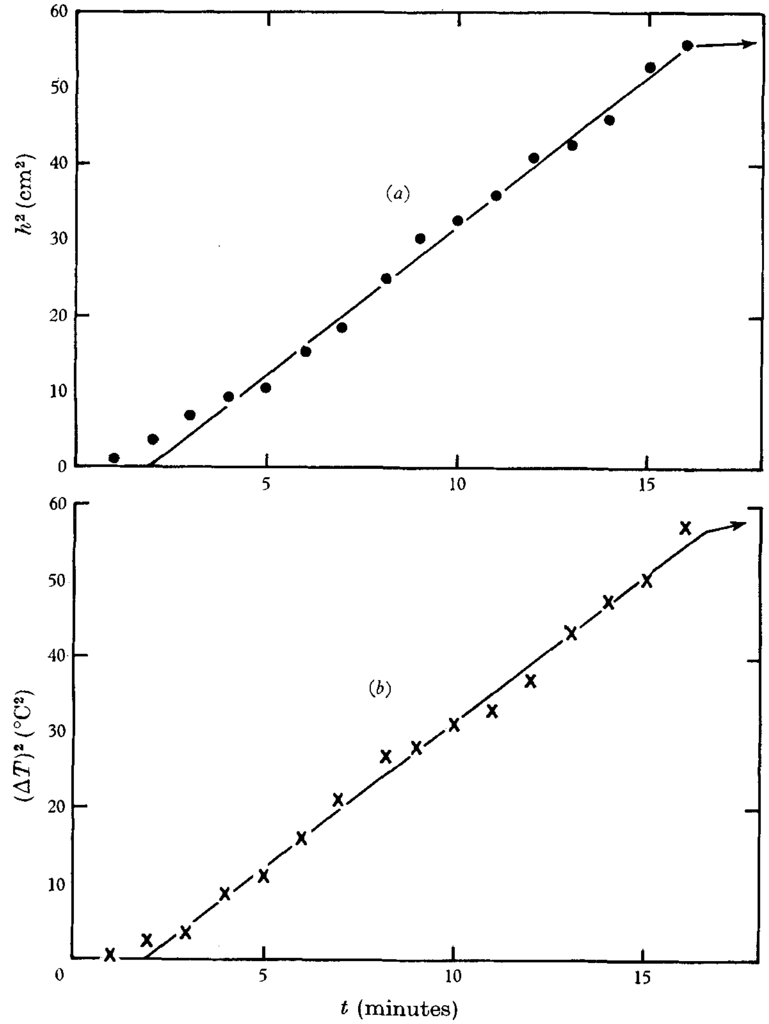

FIGURE 3. An example of the measurements on a growing convecting layer, with

$$S_* = 1{\cdot}8\times10^{-1}\,\mathrm{s}^{-2} \quad\text{and}\quad H_* = 1{\cdot}4\times10^{-2}\,\mathrm{cm}^2\,\mathrm{s}^{-3}.$$

In (*a*) the square of the layer depth h and in (*b*) the square of the temperature difference ΔT are plotted against time.

The measurements from a single run, chosen for illustration because it included a wide range of parameters and many individual readings, are plotted in figure 3. The squares of the depth and the temperature difference are well represented as linear functions of time, as suggested by (6) and (7). Notice however that the line of best fit does not pass through the origin; this reflects the fact that heating of the fluid at a steady rate did not begin immediately the heater elements were switched on, which was the point taken as $t = 0$. Some of the non-linearity near the origin must also be due to the finite time needed to form a convecting layer once heating has begun. When the virtual origin of time has been established in this way, plots of h and ΔT against t on logarithmic scales also strongly support the $t^{\frac{1}{2}}$ dependence for both these quantities.

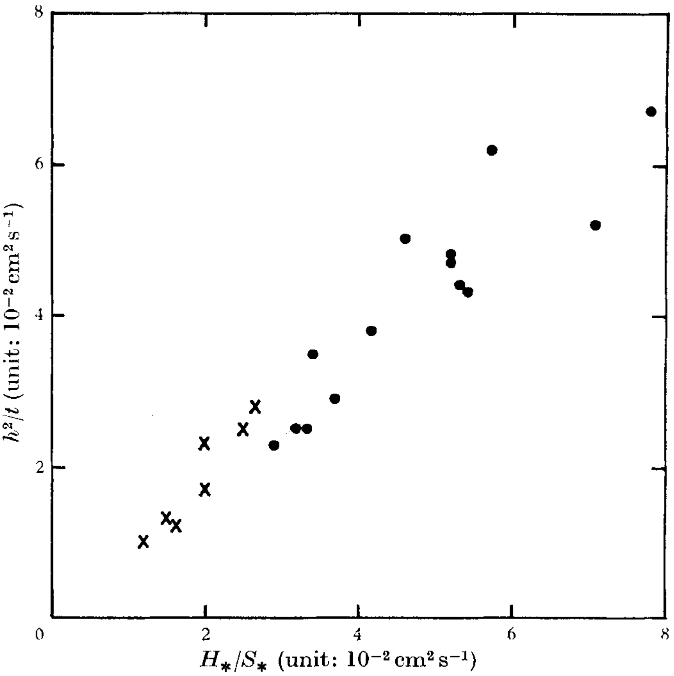

FIGURE 4. A test of equation (6): experimental values of h^2/t are compared with H_*/S_* calculated from the known heating rate and the initial salinity gradient. The two symbols are related to the effect of molecular diffusion which has been neglected in this section: they will be defined later in the caption to figure 6.

The slope of the line of figure 3a of course also allows one to make an individual numerical comparison between theory and experiment. As measured, it is $6{\cdot}7 \times 10^{-2}\,\mathrm{cm^2\,s^{-1}}$ and the predicted value from (6) is $H_*/S_* = 7{\cdot}8 \times 10^{-2}\,\mathrm{cm^2\,s^{-1}}$, using the known initial salinity gradient and heating rate. This latter ratio is also a measured quantity with its own errors, and the 15 % difference between the two values is not significant. This same procedure has been carried out for many such experiments, and the results are plotted in figure 4. If one regards each point as an independent measure of the constant of proportionality in (6) then the mean value is 0·89, with standard error ± 0·03, i.e. the measured rate of growth of the square of the layer depth as a function of time is 11 % less than predicted by (6) using H_* and S_*. This difference is statistically significant, and can probably be attributed to the escape of some heat from the top of the layer, an effect which has not yet been considered. Notice that the assumption (5) has been fully justified by these experiments; if entrainment were important the discrepancy would be in the other direction, with the constant becoming as large as 3 if the relation (4) had been used instead of (5).

For most runs, graphs of h against ΔT were also drawn as an extra check on the consistency of the results and as a simple means for extracting numerical values. The scatter about a straight line was usually greater in this form because both plotted quantities were then subject to independent errors of measurement. One obtains in this way a direct comparison between the magnitudes of the density steps due to salinity and temperature. The mean difference between them was about 10 % (which is barely significant), with a tendency for $\alpha\Delta T$ to be greater than $\beta\Delta S$, i.e. for the layer (assumed well mixed in heat and salt) to be slightly lighter than the fluid above it. Thus again there is no evidence that entrainment is important.

3. A theory for the formation of a second layer

3.1. *The diffusion of heat ahead of the advancing interface*

So far it has been assumed that all the salt and heat in the well-mixed bottom layer are confined below a sharp interface, where the properties change discontinuously. On this basis it has been shown that the depth of the layer and the temperature and salinity steps at its top all increase like $t^{\frac{1}{2}}$, and these predictions are in agreement with the observations. When one considers the criterion for formation of a second layer, however, it is essential to take the molecular processes into account. The diffusion of salt will again be ignored, but in this section the temperature structure ahead of a front, whose position and temperature are changing with time in the manner described above, will be examined.

It is helpful first to consider the diffusion equation relative to the position of a front moving vertically with a *fixed* velocity w (taken as positive). This is

$$\frac{\partial T}{\partial t} - w\frac{\partial T}{\partial z} = \kappa\frac{\partial^2 T}{\partial z^2}, \tag{9}$$

and if also the temperature at the front $z = 0$ is held constant at $T = T_0$, (9) has the solution

$$T = T_0 \exp(-wz/\kappa), \tag{10}$$

independent of time t. When w is *varying* as it does in the problem of interest here, like $w = Bt^{-\frac{1}{2}}$, say, where B is a constant, one can avoid the difficulties associated with a transformation to a non-uniformly moving frame of reference by solving the ordinary one-dimensional diffusion equation (relative to fixed axes) with the boundary conditions

$$\left.\begin{aligned} T &= At^{\frac{1}{2}} \quad \text{on} \quad \zeta = 2Bt^{\frac{1}{2}} \\ \text{and} \qquad T &\to 0 \quad \text{at} \quad \zeta = \infty, \quad \text{all}\, t \end{aligned}\right\} \tag{11}$$

(where A is another constant, and $\zeta = z + 2Bt^{\frac{1}{2}}$ is the distance from the bottom). Following the notation of Carslaw & Jaeger (1947, p. 45) the solution is

$$T = \frac{At^{\frac{1}{2}} i\,\mathrm{erfc}\,(\zeta/2(\kappa t)^{\frac{1}{2}})}{i\,\mathrm{erfc}\,(B/\kappa^{\frac{1}{2}})}. \tag{12}$$

In the limit where $B^2/\kappa \gg 1$, i.e. the velocity of advance of the front is much larger than a diffusion velocity, (12) may be written in the form

$$T = At^{\frac{1}{2}} \exp\left[-\frac{Bz}{\kappa t^{\frac{1}{2}}}\right] \exp\left[-\frac{z^2}{4\kappa t}\right] \left(1 - \frac{z}{Bt^{\frac{1}{2}}} + \text{smaller terms}\right). \tag{13}$$

If also one is interested in the behaviour at small distances z from the front and such large times that both $z/Bt^{\frac{1}{2}}$ and $z^2/4\kappa t$ are small, then (13) simplifies to

$$T = At^{\frac{1}{2}} \exp(-Bz/\kappa t^{\frac{1}{2}}). \tag{13a}$$

Thus the solution near the advancing front is again of the exponential form (10), with the instantaneous temperature and velocity replacing the constant values used earlier. How well the conditions of the experiments satisfy the above restrictions will be examined later, but for the present the argument will be continued using (13a), without further comment about its limits of validity.

3.2. *The relation between the diffusive and the convecting layers*

The solution (13a) can now be used to express certain properties of the diffusive boundary layer in terms of the parameters S_* and H_* which have been used to describe the growth of the bottom layer. The temperature gradient at the edge $z = 0$ is, from (13a), $(\partial T/\partial z)_{z=0} = -AB/\kappa$, independent of the time, and in fact the whole shape of the temperature profile remains constant with time, with the depth of penetration proportional to the temperature difference. Comparing the assumed forms of velocity and temperature variation with (6) and (7) this gradient can be put in the form

$$\left(\frac{\partial T}{\partial z}\right)_{z=0} = \frac{H_*}{2g\alpha\kappa}; \tag{14}$$

it therefore depends only on the heating rate, not the salinity gradient. A convenient (but arbitrary) measure of the 'thickness of the boundary layer' δ is defined by $\Delta T/\delta = -\frac{1}{2}(\partial T/\partial z)_{z=0}$, which implies that the exponential profile has been replaced by a linear distribution containing the same quantity of heat. With this definition the ratio of δ to the layer depth h is (from (6) and (7))

$$\delta/h = 4(\kappa S_*/H_*), \tag{15}$$

constant throughout an experiment (and already available from the analysis leading to figure 4). The ratio of the heat contained in the developing boundary layer ahead of the well-mixed layer to that retained in the layer itself is just half the ratio (15).

The criterion for the validity of the exponential solution can now be re-examined in this notation. Putting B in terms of H_* and S_* one sees that

$$B^2/\kappa = H_*/4\kappa S_* = h/\delta;$$

so the condition that B^2/κ should be large is entirely equivalent to the condition that δ/h should be small compared with 2. For the experiments to be reported here, δ/h lies between 0·054 and 0·48. At the upper end of this range, where nearly 20 % of the total heat is diffusing ahead of the growing layer, some modification of the exponential profile (and of the behaviour of the growing layer) may possibly be detectable, but for most of the experiments the argument used above should be valid.

3.3. *The criterion for breakdown*

The mechanism for instability which leads to the formation of a second convecting layer can now be discussed. The physical picture emerging from the previous section is this: for a given salinity gradient and heating rate, a destabilizing temperature difference which increases with time is being applied over a depth δ which is also increasing. Eventually, a parameter having the form of a Rayleigh number, and depending on the initial salinity gradient and the instantaneous values of ΔT and δ, must reach a critical value and convection will begin. The appropriate form of the stability criterion to adopt is of course the main question still to be resolved.

Veronis (1965) has treated the case where constant linear gradients are supposed to exist over a deep region of fluid between free boundaries. He has shown, as mentioned earlier, that in the case where the salinity is stabilizing and the temperature destabilizing, this system should first become unstable (to infinitesimal disturbances) in an overstable mode. The criterion for onset of convection is

$$R = \frac{\nu}{\nu+\kappa}R' + \tfrac{27}{4}\pi^4, \tag{16}$$

where

$$R = \frac{g(\Delta\rho)_T d^3}{\rho\nu\kappa}, \quad R' = \frac{g(\Delta\rho)_S d^3}{\rho\nu\kappa}. \tag{17}$$

Note particularly that κ, the thermometric conductivity for *heat* appears in both R and R', which otherwise have the form of Rayleigh numbers based on density differences due to heat and salt and on the layer depth d. When R' is large, i.e. $R' \gg \frac{27}{4}\pi^4$, the system as a whole breaks down just before the hydrostatic density distribution becomes heavy at the top. This is very different from the prediction for the growing mode, to obtain which the unstable density gradient due to heat must become substantially greater than the stabilizing gradient due to salinity.

One would expect that the same ideas would carry over, qualitatively at least, to the case of interest here, where not only the temperature difference is increasing as the limit of stability is approached but also the depth over which it is applied. The main difference from Veronis's original interpretation arises because it can no longer be assumed that R' is large. For if the depth d in (17) is identified with the characteristic thickness δ used in (15),

$$\left.\begin{aligned} g(\Delta\rho)_S/\rho &= 2S_*\,\delta, \\ \text{whereas} \qquad g(\Delta\rho)_T/\rho &= S_* h. \end{aligned}\right\} \tag{18}$$

Thus $R'/R = 2\delta/h$, which has been shown in the last section to be small for the cases of interest here. As the boundary layer grows in a given experiment, R' will increase from zero but remain a fixed fraction of R, and the condition for breakdown is therefore that the ordinary Rayleigh number R, based on heat alone, should exceed a critical value. For small δ/h this value will be little different from the critical Rayleigh number R_c which applies when no salt is present. This dominance of heat over salt arises because the Rayleigh number depends mainly on the thickness δ of the boundary layer. This is determined by the diffusion of heat, and the whole step of temperature is applied across δ, whereas the corresponding salinity difference is that due to the original linear gradient, almost unchanged by diffusion. (In the opposite limit, when the heating rate is so small that the diffusion velocity for heat is much larger than the rate of advance of the front, (16) implies that the fluid above cannot become unstable in this way, because the temperature gradient remains small compared with the initial salinity gradient. Intermediate conditions require a more careful discussion.)

The final step of this argument is the expression of the critical Rayleigh number in terms of the measurable parameters h, H_* and S_*. Substituting for δ from (15) in (17) gives

$$R = -\frac{64\kappa^2}{\nu} g\alpha\Delta T\left(\frac{S_* h}{H_*}\right)^3. \tag{19}$$

Using (6) and (7) to eliminate ΔT, and rearranging, leads to an expression for the 'critical depth' h_c (the maximum depth of the bottom layer before a second forms above it) in terms of R_c

$$h_c = \left(\frac{\nu R_c}{64\kappa^2}\right)^{\frac{1}{4}} H_*^{\frac{3}{4}} S_*^{-1}. \tag{20}$$

It is also convenient to have this relation in terms of the flux of density deficit $\rho_* = H_*/g$ and the initial density gradient due to salt $d\rho_s/dz = 2\rho S_*/g$

$$h_c = \left(\frac{\nu R_c}{4g\kappa^2}\right)^{\frac{1}{4}} \rho_*^{\frac{3}{4}} \left(\frac{1}{\rho}\frac{d\rho_s}{dz}\right)^{-1}, \tag{21}$$

since it is in this form that it will be tested experimentally.

It is impossible to write down a definite numerical value for R_c, since the stability problem has not been solved for exponential profiles, nor do we know exactly what boundary conditions to apply. For a linear profile and with a quasi-static assumption, $R_c \approx 10^3$, but there remains the question of the replacement of a non-linear profile by a linear one. There are several relevant, but inconclusive, studies which should be mentioned. Howard (1964) has deduced a value for R_c for an error function profile, which is of the same order of magnitude as the ordinary criterion when the depth scale is defined appropriately. Currie (1967) has carried out a linear stability analysis for the time-dependent case, representing a non-linear profile by an 'equal area' straight line equivalent (exactly the assumption which was used in §3.2 to define δ). His analysis suggests that R_c, based now on the total depth, can vary from slightly less than the usual theoretical value to several times this as the heating rate is increased, in agreement with the experimental result of Soberman (1959).

The uncertainties inherent in applying any of these results to the case of interest are so great that no numerical value will be assumed. Instead, the experimental results will be used to test the predicted functional form (21), and the main check on the theory will be the adequacy of this in representing the power law dependence on ρ_* and $d\rho_s/dz$. A mean experimental value of the constant of proportionality will also emerge, however, and the corresponding value of R_c should not be too different from that suggested in other contexts if it is to be claimed that there is good agreement between theory and experiment.

4. The formation of a second layer: experiments

4.1. *The oscillatory breakdown*

The arguments of §3.3 have depended on the assumption of the overstable criterion (16) as the relevant one for instability. The results of Shirtcliffe (1967) have been quoted to give some experimental support to this, but it seemed desirable to attempt a further direct confirmation under the conditions of the present experiment. Measurements were made using thermistors of the temperature at a point, and the temperature difference between two points separated by $\frac{1}{2}$ cm in the vertical, at various heights above the bottom of a linearly stratified tank of salt solution. Heating was supplied at a much smaller rate than for the main body of the experiments, so that the initial instability could be studied (rather than the growth of the layer once it had formed). An oscillatory behaviour was observed on both types of records, but we found (as did Shirtcliffe) that the phenomenon was much more clearly shown using the difference method.

With the probes $\frac{1}{2}$ cm and 1 cm from the bottom of the tank, the temperature gradient increased smoothly and steadily for a while; the record gradually became more noisy, until bursts of regular oscillations appeared and died away, as is shown in the section of record labelled *a* in figure 5. A little later, many cycles of an oscillation with nearly constant amplitude were observed, until finally the amplitude increased and the layer broke into convective motion (trace *b*). This behaviour is rather different from that reported by Shirtcliffe, who demonstrated an oscillation with steadily increasing amplitude (his figure 2), but these differences could well be due to the higher sensitivity of his recording technique (which used thermocouples). In that part of his record where the sensitivity was lower, there is an interval of nearly steady oscillations with an amplitude comparable with ours. Since the absolute values of the temperature gradient were not accurately known for the recordings reproduced in figure 5, it is not possible to check the criterion for breakdown directly against theory. The period of the oscillation is well defined, however; as measured from figure 5 it is 20 sec. This can be compared with the predictions of Veronis (1965), who showed that the period of an overstable oscillation at marginal stability should be $[3(\nu+\kappa)/\kappa]^{\frac{1}{2}}$ (i.e. about 5) times the Brunt–Väisälä period based on the initial salinity gradient, and that the factor decreases to about 3 when the horizontal wave number is large. In our experiment this latter period (calculated from the mean linear salinity profile) is 5·7 sec; the measured value is indeed several times this, an agreement which is as good as can be expected in view of the great differences between the experimental conditions and those assumed in the theory.

Similar recordings of temperature gradient were made with the probes just ahead of a growing layer, in order to study the breakdown of the second layer which is of direct concern here. There was now the extra difficulty that fluctuations of the measured gradient were also produced by the oscillations of the interface due to the convection in the layer below. On several records, however, such as that reproduced in figure 5(*c*), there is clear evidence of intervals of regular oscillations with the period characteristic of the overstable mode, superimposed on slower, more irregular motions. This therefore does seem to be the mechanism of instability, whether the lower boundary is moving or not.

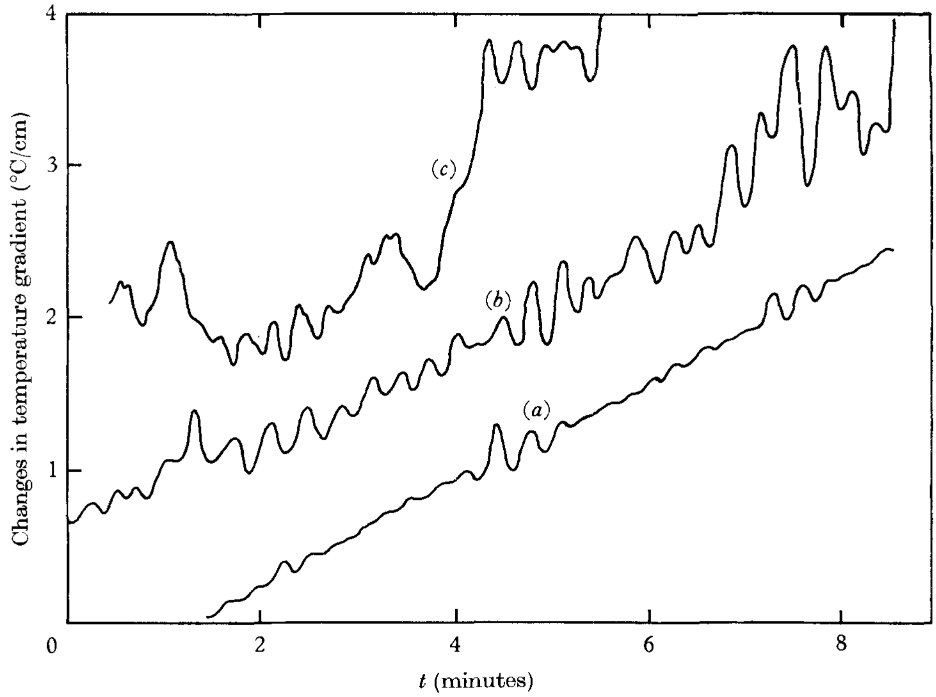

FIGURE 5. Fluctuations in temperature gradient recorded with two thermistors, when a linearly stratified salt solution was heated from below. Only differences in the gradients are relevant, and the records have been shifted arbitrarily along both axes for convenience of reproduction.

(a) Thermistors $\frac{1}{2}$ cm and 1 cm from the bottom of the tank. Bursts of oscillations appear and then die out again.

(b) This record is from the same run as (a) with a gap of about one minute. Nearly steady oscillations eventually grow in amplitude and become irregular.

(c) Thermistors $8\frac{1}{2}$ cm and 9 cm above the bottom of the tank, ahead of an advancing interface. Similar oscillations to those in (b) are seen, superimposed on larger slower disturbances.

4.2. *Experimental determination of layer depths*

The experiments described in §§2.2 and 2.3 were continued until the bottom convecting layer stopped growing and a second formed above it; the same data, with the addition of this maximum depth, are available to test the theory of §3. In fact rather more experiments were suitable for this purpose than were used in figure 4, since the final layer depth was well defined even in cases where its rate of growth was irregular and hard to measure.

Again there is a wide choice of methods of presentation; two have been chosen which test the several predictions contained in (21). The power law dependence is most easily checked by plotting $(h_c/\rho)(d\rho_s/dz)$, which is proportional to the density step at the top of the layer when it forms, against the heating rate (or ρ_*)

on logarithmic scales. This plot is shown in figure 6. A line of slope $\frac{3}{4}$ is drawn through the points; to within the accuracy of measurement, the experimental results are well described by an equation of the form (21). In order to assess the possible effects of deviations from the limit of small δ/h (assumed in § 3.1), two different symbols have been used in figure 6 (also in figure 7, and earlier in figure 4). A division has been made (arbitrarily) at a value of $\delta/h = 0{\cdot}20$; dots denote experiments with $\delta/h < 0{\cdot}20$, and crosses, $\delta/h > 0{\cdot}20$. Though several of the points with large deviations from the line drawn in figure 6 do correspond to the larger

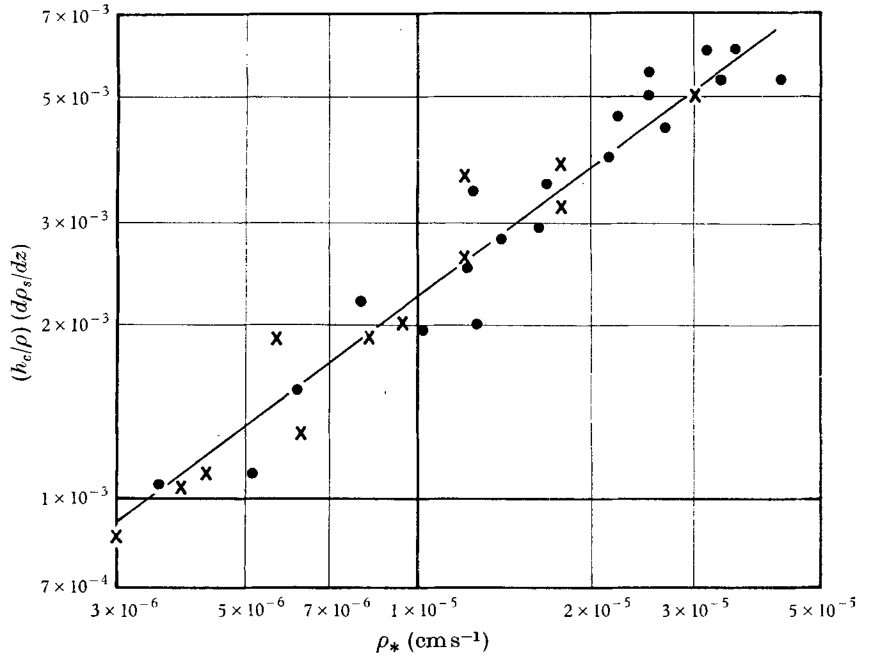

FIGURE 6. The product of the measured maximum depth of the bottom layer h_c and the density gradient due to salinity $(d\rho_s/dz)$, plotted against the density flux due to heating, on logarithmic scales. The symbols denote different ranges of values of δ/h; ● $\delta/h \leqslant 0{\cdot}2$, × $\delta/h > 0{\cdot}2$. A line with slope $\frac{3}{4}$, suggested by equation (21), is drawn through the points.

δ/h, it is clear that the experiments are not accurate enough to investigate this effect in detail, and that the simple theory gives an adequate description to the present accuracy. The same conclusion applies to figure 4; the experiments with the larger δ/h (those with small heating and growth rates) do not deviate significantly from the line which can be drawn through the rest of the points.

To test the quantitative predictions of (21), it is convenient to plot $(h_c/\rho)(d\rho_s/dz)$ against $\rho_*^{\frac{3}{4}}$ on linear scales, and this has been done in figure 7, with all the physical constants (evaluated at the appropriate temperature) included with $\rho_*^{\frac{3}{4}}$ so that both scales are non-dimensional. The slope of the line drawn to fit these points is $R_c^{\frac{1}{4}} = 12{\cdot}5$ with standard error $\pm 0{\cdot}3$. This implies a value of $R_c \approx 2{\cdot}4 \times 10^4$, an order of magnitude larger than the simplest extrapolation of existing theory would suggest. All the corrections to (13a) which would be introduced by using (13) or by relaxing the condition $B^2/\kappa \gg 1$ will tend to reduce the effective thickness of the boundary layer and hence also reduce R_c. The measured maximum layer depth is, however, predicted to better than a factor of two, with any reasonable choices of the definition of δ and the numerical value of R_c, and without the need to introduce any further empirical constant.

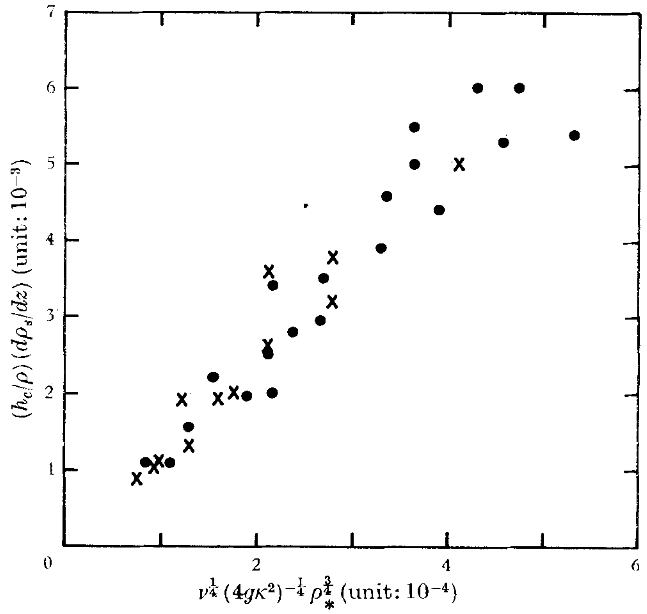

FIGURE 7. The same data as in figure 6, plotted now on linear scales, with $\rho_*^{\frac{3}{4}}$ and all the physical parameters grouped along the abscissa. The symbols have the same meaning as before.

5. Discussion

In conclusion, the results obtained above will be fitted into the general picture of what is now known about transfers in a liquid containing both temperature and salinity differences, and the new questions raised by them will be discussed further. The understanding of the growth of the first layer described in § 2 is in a satisfactory state; experiment agrees with the theory based on a negligible extra entrainment at the top of the layer due to the conversion of turbulent kinetic to potential energy. There does, however, remain the doubt (as with all laboratory convection experiments) whether this would continue to be true on a much larger scale. The criterion which determines the maximum (critical) layer depth at which a second layer forms above the first also seems to be satisfactory. One should emphasize here the good agreement between the theoretical and measured functional forms; the discrepancy between the numerical constants merely points to the need for further development of time-dependent stability theory. At the very least, the results obtained here should remove any doubts about the importance of bottom heating compared with that from the sides during the growth of the lowest layer.

All the ingredients are now available which makes it possible to study the growth of a whole series of layers above the first. Experimentally it may be difficult to eliminate entirely, in the laboratory, the effects of side-wall heating (as discussed earlier). On a geophysical scale the one-dimensional case will certainly be relevant, however, and theoretically the way is clear. Given the heating rate and salinity gradient, the present work can be used to determine when a new layer will form. The rates of transfer of heat and salt between the layers already formed, and hence the net rate of heating affecting the topmost interface, can be calculated using the previous set of experimental results (Turner 1965). It seems to be a matter for straightforward computation to reproduce the observed sequence of events: layer formation at the top, and the mixing of the bottom two layers when the density difference between them becomes small. It would be useful to discover if a quasi-steady state could be achieved, and if so, the number of layers which can be maintained in various circumstances and the mean temperature and salinity gradients associated with them.

Finally, some remarks can be made about the related situation of a stable temperature gradient made unstable by adding salt at the top. In this case, the vertical transport takes place in long narrow convection cells ('salt fingers') across interfaces between turbulent convecting layers. The fluxes of salt and heat across established interfaces of this kind have been measured in the laboratory and the results applied to oceanic observations (Turner 1967). An important point made qualitatively in the paper quoted is that, so far as the formation of layers from a gradient is concerned, the salt finger case and the one studied in the present paper are entirely symmetrical: the basic physical parameter in both is a net unstable buoyancy flux. Now it is possible to go further and suggest a criterion for the maximum depth of the first layer formed as salt is added above a temperature gradient. A similar argument to that used in § 3, which will not be reproduced here, indicates that a relation of the form (21) will again apply, but with the roles of heat and salt just interchanged in the initial density gradient and buoyancy flux terms. A numerical constant (equivalent to R_c) remains to be determined, either by experiment or from a more detailed theory, and further discussion of this point will be deferred until more definite results can be reported.

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REFERENCES

Ball, F. K. 1960 Control of inversion height by surface heating. *Q. J. R. Met. Soc.* **86**, 483–494.

Carslaw, H. S. & Jaeger, J. C. 1947 *Conduction of heat in solids.* Oxford University Press.

Currie, I. G. 1967 The effect of heating rate on the stability of stationary fluids. *J. Fluid Mech.* **29**, 337–347.

Hoare, R. A. 1966 Problems of heat transfer in Lake Vanda, a density stratified Antarctic lake. *Nature, Lond.* **210**, 787–789.

Howard, L. N. 1964 Convection at high Rayleigh number. *Proceedings 11th International Congress of Applied Mechanics, Münich*, ed. H. Görtler. Berlin: Springer-Verlag.

Lilly, D. K. 1967 Models of cloud layers under a strong inversion. *NCAR* manuscript no. 386 (to be published)

Oster, G. 1965 Density gradients. *Scient. Am.* **213**, 70–76.

Sani, R. L. 1965 On finite amplitude roll cell disturbances in a fluid layer subjected to a heat and mass transfer. *Am. Inst. Chem. Engng J.* **11**, 971–80.

Shirtcliffe, T. G. L. 1967 Thermosolutal convection: observation of an overstable mode. *Nature, Lond.* **213**, 489–490.

Soberman, R. K. 1959 Onset of convection in liquids subject to transient heating from below. *Phys. Fluids*, **2**, 131–138.

Swallow, J. C. & Crease, J. 1965 Hot salty water at the bottom of the Red Sea. *Nature, Lond.* **205**, 165–166.

Turner, J. S. 1965 The coupled turbulent transports of salt and heat across a sharp density interface. *Int. J. Heat Mass Transfer*, **8**, 759–767.

Turner, J. S. 1967 Salt fingers across a density interface. *Deep Sea Res.* **14**, 599–611.

Turner, J. S. & Kraus, E. B. 1967 A one-dimensional model of the seasonal thermocline. *Tellus*, **19**, 88–97.

Turner, J. S. & Stommel, H. 1964 A new case of convection in the presence of combined vertical salinity and temperature gradients. *Proc. U.S. Nat. Acad. Sci.* **52**, 49–53.

Veronis, G. 1965 On finite amplitude instability in thermohaline convection. *J. Mar. Res.* **23**, 1–17.